

Vacuum Structures of Supersymmetric Noncompact Gauge Theory

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Abstract

We consider models with a noncompact symmetry in the framework of $\mathcal{N} = 1$ supersymmetry. Contrary to the conventional approach, the noncompact symmetry is realized linearly on all fields without constraints. The models are constructed using noncanonical Kähler function and gauge kinetic function, which is introduced for the local case. It is explained that the symmetry needs to be spontaneously broken for the consistency of a model. We study the vacuum structures of two models with the noncompact symmetry $SU(1,1)$ for both global and local cases. One of them includes two fundamental representations of the group and the other includes one adjoint representation. It is shown that the former is consistent for the global case and the latter is consistent for both the global and local cases.

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I. INTRODUCTION

Two of the present authors have studied a supersymmetric vectorlike model based on $\mathcal{N} = 1$ supersymmetry and a horizontal symmetry G_H [1–3], which governs the generational structures of quarks and leptons [4–9]. It is constructed to reproduce the (constrained) minimal supersymmetric standard model (MSSM) [10–14] at low energies. This model seems to be a very promising candidate for giving the explanation of the physics beyond the standard model and the MSSM. It has several distinguishing features: The appearance of three chiral generations of quarks and leptons observed in Nature is explained as a result of a dynamical phenomenon, spontaneous breaking of the noncompact horizontal symmetry $G_H = SU(1, 1)$. It also naturally explains the hierarchical structure of the Yukawa couplings as a consequence of the symmetry property of the group. Furthermore, the model gives rise to the violation of P , C and T symmetries observed in experiments also as a consequence of the spontaneous breaking of $SU(1, 1)$ gauge symmetry, while being the exact symmetries of the model. In this scenario, the noncompact gauge symmetry plays an extremely important role.

Although the model has several intriguing features, there remain some important elements that need clarification. One of them concerns the issue of whether a gauge theory based on a noncompact group can be constructed at all. Most of the studies of gauge theories so far are devoted to the ones with compact groups and many of the important properties have been understood. In contrast, it seems that only a little is understood about theories with noncompact internal symmetry groups. Among them, it is known that theories based on a noncompact group such as $SU(1, 1)$, global or local, are potentially afflicted with so-called the “ghost” problem [15], which could lead to the breakdown of such theories.

To explain the problem briefly, let us consider the gauge fields associated with $SU(1, 1)$, which belong to the adjoint representation of the group. The “canonical” kinetic term for the fields is given as

$$\mathcal{L}_{\text{gauge kin.}} = \frac{1}{4} \eta_{AB}^{(3)} F_{\mu\nu}^A F^{\mu\nu B}, \quad (1)$$

where the indices A and B run over the adjoint representation and $\eta_{AB}^{(3)}$ is the metric of

$SU(1, 1)$ in the adjoint representation(or Killing form) defined as

$$\eta_{AB}^{(3)} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}, \quad (2)$$

and $F_{\mu\nu}^A$ are the field-strength tensor. We see immediately that the metric in the kinetic term is not positive-definite. The third component ($A = 3$) has a sign opposite to the first and second ones ($A = 1, 2$), which have the standard sign. It should be emphasized that the actual problem is the simultaneous occurrence of the different signs. This kind of kinetic term alone would give rise to the perturbative quanta with negative norm, which we refer to as a “ghost” in this paper, and thus we end up with the Fock space with an indefinite metric [34]. In such theories, unitarity is violated, and therefore we would have difficulty in making probabilistic interpretations in quantum theory, at least in perturbation theory. Another problem is that the Hamiltonians may be unbounded from below. If a ghost appears in a theory, the energy of the system may decrease arbitrarily as more ghosts are created, and thus its appearance implies the absence of a ground state. One might notice that these two problems are not independent. If a ground state does not exist, one can not define any other states. Hence, it would not make much sense to dispute about the violation of unitarity to begin with. The situations are basically the same for fields of other spin and in other representations of the group. Therefore, we would encounter the same problem also in the case of a global symmetry.

In the case of a global symmetry, a prescription to construct a theory based on a noncompact group, which is now a standard one, is worked out and explained in Ref. [16]. In this construction, a noncompact global group G is spontaneously broken to its maximal compact local subgroup H so that G is realized nonlinearly while H is realized linearly (hidden local symmetry [17]). The scalar fields in the theory correspond to Nambu-Goldstone(NG) bosons that parametrize the coset space G/H , i.e. nonlinear sigma model. The essential point here is that the non-propagating “gauge” fields associated with local group H exactly cancel the kinetic terms for the scalar fields that correspond to the ghosts. In other words, the apparent ghost degrees of freedom are not dynamical and simply absent right from the beginning. There are many studies along this line on the nonlinear sigma models based on noncompact group including the evaluation of quantum corrections [18–25].

In the case of a noncompact local group, the ghosts may appear in gauge fields as well as in fields of other spin. In Ref. [26], de Wit et al. study such case in the framework of $\mathcal{N} = 2$ supergravity. In order to avoid the appearance of ghosts, they propose to use a nonlinear multiplet of the group and to use the field that is a part of the multiplet as a compensator. An important point in their construction is the use of the framework of $\mathcal{N} = 2$ supergravity that is reduced from the conformally invariant one by imposing certain gauge conditions [27]. The compensating field is extended to a supermultiplet, which contains a field that corresponds to a ghost. After the elimination of the tensor auxiliary field, the sign of the kinetic term for the gauge fields is reversed and the ghost disappears from the theory.

In this paper, we explore the possibility of constructing a sensible theory with a *linearly* realized noncompact *local* group in the framework of $\mathcal{N} = 1$ global supersymmetry. The basic idea resembles that of Ref. [26]. However, there are essential differences: We consider a theory with $\mathcal{N} = 1$ supersymmetry. A noncompact symmetry is realized linearly with all fields being independent dynamical degrees of freedom, i.e. without imposing the constraints that reduce the degrees of freedom, which are used to define nonlinear sigma models. Our construction does not need to involve auxiliary fields nor compensators, which play essential roles in the model studied in Ref. [26] [35]. We show that a gauge theory based on a noncompact group can be defined as a sensible theory by presenting an explicit model. In this model, the symmetry is realized linearly on all the unconstrained fields. We also study the structure of the vacuum that are far more richer than those in nonlinear sigma models and its relation to the ghost problem.

We briefly sketch out the scenario for constructing a ghost free theory with a noncompact group in both global and local cases. To be specific, the ghost problem arises in theories with nonunitary representations of a noncompact group, e.g. finite-dimensional representations. In this light, we focus our attentions on theories with nonunitary representations. The essential points in the scenario are the introduction of field-dependent kinetic terms for the fields and the necessity of the spontaneous breaking of the noncompact group. This seems to be the only possibility for realizing a sensible theory of this kind, which means that a simple free theory can not be defined in the case of a noncompact group.

Suppose that there exists a scalar field ϕ that belongs to some representations of the

group. The gauge invariant kinetic term for the gauge fields is given as

$$\mathcal{L}_{\text{gauge kin.}} = -\frac{1}{4}f_{AB}(\phi)F_{\mu\nu}^A F^{\mu\nu B}, \quad (3)$$

where the indices A and B run over the adjoint representation of the group, i.e. $A, B = 1, 2, 3$, and $f_{AB}(\phi)$ is a symmetric tensor that is a function of scalar field ϕ . It is important to remember that, in quantum field theories, fluctuations of a field around the vacuum, i.e. the ground state, are identified as excitations of corresponding particle. Accordingly, we should consider an expansion of fields around the vacuum to treat perturbative quanta. If the function $f_{AB}(\phi)$ acquires nontrivial vacuum expectation value (VEV) and all of its eigenvalues are positive-definite no ghosts appear in the gauge fields. It is plausible to expect that the similar mechanism eliminates the ghosts in other fields as well.

Now, we see that a Lagrangian of a ghost free theory necessarily contains nonrenormalizable terms. However, this should not be a problem if we consider the theory to be an effective theory of more fundamental theory, which is perhaps the case for any quantum field theories of phenomenological interest. Rather, the presence of nonrenormalizable terms in the theory merely implies the fact that there is unknown short-distance structure that is not treated explicitly and there is no theoretical difficulty [28, 29]. For the reasons that we explained at the beginning of this section and for the fact that the analysis becomes simple, we consider a construction in the the framework of $\mathcal{N} = 1$ global supersymmetry.

The structure of the paper is as follows. In section II, we review and introduce materials that are necessary in the analysis of the models. In section III, we present two models based on the global $SU(1, 1)$, one with two doublets and the other with a single triplet in order to explain the problems and the conditions for the superpotential and Kähler potential. Also, the analysis of the vacuum structure and the discussion of the transition between degenerate vacua are presented. In section IV, we consider the gauging of the symmetry and examine the necessary conditions for the gauge kinetic function. It is found that, in the case of the local symmetry, two doublet model considered in this paper is inconsistent. Some remarks about the obstacles in the construction of the model with noncompact symmetry are given. Section V is devoted to the summary and discussions of the prospects for the extension of the model.

II. BASIC SETUP

In this section, we review the basic elements of our framework in order to elucidate what must be achieved in order to construct a consistent theory based on a noncompact group. We also briefly review the transformation properties of the representations of $SU(1,1)$, which are useful for construction of invariants.

A. $\mathcal{N} = 1$ global supersymmetry

The Lagrangian of general $\mathcal{N} = 1$ globally supersymmetric theory with terms up to and including two spacetime derivatives is completely determined by specifying three functions; the superpotential W , the Kähler potential K , and the gauge kinetic function f_{AB} . The gauge kinetic function is introduced only when there is an internal local symmetry. The superpotential $W(\Phi)$ and the gauge kinetic function $f_{AB}(\Phi)$ are *arbitrary* functions that are holomorphic in the chiral superfield (there could be more than one), collectively denoted by Φ , and have mass dimensions three and zero respectively. The Kähler potential $K(\Phi, \Phi^\dagger)$ is also an *arbitrary* real function of chiral and anti-chiral superfields, and has mass dimension two. The mass dimensions here refer to the ones in four-dimensional spacetime ($D = 4$). Note that both the superpotential and the Kähler potential must be invariant under the symmetry transformation, while the gauge kinetic function must be constructed so that it transforms as a symmetric product of two adjoints of the gauge group.

The kinetic terms for chiral multiplets originate from a Kähler potential, while those for vector multiplets come from a gauge kinetic function. The bosonic part of the Lagrangian is given as follows

$$\mathcal{L}_{scalar} = K^{\bar{i}j} (D_\mu \Phi)_{\bar{i}}^\dagger (D_\mu \Phi)_j - V_F, \quad (4)$$

$$V_F = W^i (K^{-1})_{i\bar{j}} W^{\bar{j}}, \quad W^i \equiv \frac{\partial W}{\partial \Phi_i}, \quad W^{\bar{i}} \equiv (W^i)^\dagger, \quad (5)$$

$$\mathcal{L}_{gauge} = -\frac{1}{8} \left(f_{AB} + (f_{AB})^\star \right) F_{\mu\nu}^A F^{B\mu\nu} + \frac{i}{8} \left(f_{AB} - (f_{AB})^\star \right) F_{\mu\nu}^A \tilde{F}^{B\mu\nu} - V_D, \quad (6)$$

$$V_D = \frac{g^2}{4} \left\{ (f^{-1})^{AB} + (f^{-1})^{\star AB} \right\} \left(K^i (H_A)_i{}^j \Phi_j \right) \left(K^l (H_B)_l{}^m \Phi_m \right), \quad K^i \equiv \frac{\partial K}{\partial \Phi_i}, \quad (7)$$

where D_μ is a covariant derivative for a corresponding representation, g is the gauge coupling constant, and H_A are generators in a corresponding representation. We have also introduced

the Kähler metric $K^{\bar{i}j}$; a dimensionless quantity defined as

$$K^{\bar{i}j} \equiv \frac{\partial^2 K}{\partial \Phi_i^\dagger \partial \Phi_j}. \quad (8)$$

The inverses of the Kähler metric and gauge kinetic function are defined by the following

$$K^{\bar{i}j} (K^{-1})_{j\bar{k}} = \delta^{\bar{i}}_{\bar{k}}, \quad f_{AB} (f^{-1})^{BC} = \delta_A^B. \quad (9)$$

The indices i, j, k and $\bar{i}, \bar{j}, \bar{k}$ run over all scalar fields in the corresponding representations and the indices A and B run over the adjoint representation of the group. The bars on the indices represent the conjugates. The raising and lowering of the indices are to be done using the metric of the symmetry group in the corresponding representations and the upper and lower indices (with and without bars) are to be contracted with each other in the standard manner to form invariants. It should be understood that only the bosonic components of superfields are to be retained in the formula. The terms denoted by V_F and V_D are the scalar potentials coming from the F-term and the D-term respectively.

Let us make a few comments on V_D . Using the invariance property of W and K under the symmetry transformation, we can verify that the following relations hold

$$\frac{\delta W}{\delta \Phi_i} (H_A)_i^j \Phi_j = 0, \quad (10)$$

$$\frac{\delta K}{\delta \Phi_i} (H_A)_i^j \Phi_j = \Phi_i^\dagger \left(H_A^\dagger \right)_{\bar{j}}^{\bar{i}} \frac{\delta K}{\delta \Phi_{\bar{j}}^\dagger}, \quad (11)$$

Using the relation given in eq.(11), we can confirm that V_D is Hermitian. Note that the potential V_D is present only in the case of a local symmetry. It is important to note that, in general, the Kähler metric and the gauge kinetic function are field-dependent.

As we have explained in section I, the metrics of the kinetic terms for the perturbative quanta expanded around the vacuum must be positive-definite in order for the ghosts to disappear from the theory. It is essential to find a Kähler potential that gives rise to a positive-definite Kähler metric and a gauge kinetic function, whose real part is also positive-definite at the vacuum.

As we see from eq.(7) and eq.(5), the scalar potentials depend on the real part of $(f^{-1})^{AB}$ and $(K^{-1})_{i\bar{j}}$ respectively. This is a special property of supersymmetric theories; the kinetic term and the potential term are related to each other through a single function. This has a very important consequence. If the metric of the kinetic term is of an indefinite one,

the potential would be unbounded. The consistency requires that, if the vacuum exists, the Kähler metric evaluated at the vacuum must be positive-definite. The analysis of the Kähler metric must be done for presumed VEV's, which are to be determined by minimizing the potential that itself depends on the Kähler metric. Similar argument applies to the real part of the gauge kinetic function as well. We should stress that the conditions required for the positivity of the metric and the ones to ensure the boundedness of the potential may be different.

Because we consider our theory as an effective theory, we need to identify the scales in the theory in order to define a low energy expansion. In this paper, we assume that there are two scales in our theory; a physical cutoff M_{high} , which is typically of order of the mass of the lightest degrees of freedom that is not treated explicitly and a low-energy scale M_{low} that is much smaller than M_{high} [36]. One may imagine that the low energy scale M_{low} is generated through the dynamics of yet unknown more fundamental theory. Here we just assume that such a scale exists in the theory and do not ask its origin. We also assume that all parameters in $W(\Phi)$, $K(\Phi, \Phi^\dagger)$, and $f_{AB}(\Phi)$ with positive mass dimension $d > 0$ are of $\mathcal{O}(M_{low}^d)$ and the parameters with negative mass dimension $d < 0$ are of $\mathcal{O}(M_{high}^d)$. The latter follows from the standard naive dimensional analysis[30]. We should emphasize that the assumptions here are crucial for our construction. In particular, the scaling of couplings with positive mass dimension are essential in order for our theory to be valid up to the energy scale of the cutoff M_{high} . Therefore it is very important to see whether the assumption are stable when quantum corrections are taken into account. However, it is out of the scope of the present paper and we leave this question to the future investigation.

As a consequence of supersymmetry, the kinetic terms for chiral fermions and gaugino are expressed in terms of Kähler metric and gauge kinetic function respectively. Therefore, if the ghosts in scalar fields and the gauge fields are eliminated successfully, the ghosts in fermions disappear automatically.

In order for the field-dependent part of the Kähler metric and the gauge kinetic function to play significant roles, the scalar fields must acquire nontrivial VEV's [37]. It means that the consistency requires that the noncompact symmetry must be spontaneously broken.

To summarize, in order to construct a consistent gauge theory based on a noncompact local group in a supersymmetric framework, we need to find the superpotential, Kähler potential, and gauge kinetic function that realize all of the following properties simultaneously:

i) The theory has well-defined vacua. That is, the Hamiltonian is bounded from below. ii) Only the vacua that break the noncompact symmetry are allowed. iii) The Kähler metric at the vacuum is positive definite. iv) The real part of gauge kinetic function at the vacuum is positive-definite.

B. $SU(1,1)$ symmetry

We consider the transformation properties of the fields in the fundamental and adjoint representations of $SU(1,1)$, which will be introduced in our models.

First of all, we define the generators of $SU(1,1)$ to satisfy the following relations,

$$[H_1, H_2] = -iH_3, \quad [H_2, H_3] = iH_1, \quad [H_3, H_1] = iH_2. \quad (12)$$

The algebra resembles that of $SU(2)$. The only difference between the two is the sign of the right-hand side of the first equation in eq.(12).

Let us denote a fundamental representation of $SU(1,1)$ by Φ , which is a doublet. The transformation law for a doublet is defined as

$$\Phi \rightarrow U\Phi, \quad (13)$$

where U is an element of $SU(1,1)$ in the fundamental representation and satisfies

$$U^\dagger \eta^{(2)} U = \eta^{(2)}, \quad (14)$$

where $\eta^{(2)}$ is the metric of $SU(1,1)$ in the fundamental representation defined by the following two-by-two matrix,

$$\eta^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (15)$$

From eq.(14), we notice that $U^{-1} = \eta^{(2)} U^\dagger \eta^{(2)}$. This helps us to see the following transformation law

$$\Phi^\dagger \eta^{(2)} \rightarrow \Phi^\dagger \eta^{(2)} U^{-1}. \quad (16)$$

The explicit form of U that is useful for examining the transformation properties is

$$U = a_0 \mathbf{1}_{2 \times 2} + 2i \sum_{A=1}^3 a_A H_A^{(2)}, \quad (17)$$

where a_i ($i = 0, 1, 2, 3$) are real parameters with a constraint $a_0^2 - a_1^2 - a_2^2 + a_3^2 = 1$. Here, $\mathbf{1}_{2 \times 2}$ is a two-by-two unit matrix and $H_A^{(2)}$ are the generators of $SU(1, 1)$ in the fundamental representation. Another representation of U is given in appendix A. We choose the following matrices to represent H_A ,

$$H_1^{(2)} = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H_2^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H_3^{(2)} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (18)$$

They satisfy the following normalization condition,

$$\text{tr} \left(H_A^{(2)} H_B^{(2)} \right) = \frac{1}{2} \eta_{AB}^{(3)}. \quad (19)$$

We can also confirm the relation $\eta^{(2)} H_A^{(2)\dagger} \eta^{(2)} = H_A^{(2)}$ by using the explicit expression. Given the explicit form of U , we can show the following transformation law

$$\Phi^T H_2^{(2)} \rightarrow \Phi^T H_2^{(2)} U^{-1}. \quad (20)$$

Let us denote an adjoint representation of $SU(1, 1)$, which is a triplet, by

$$X = \eta_{(3)}^{AB} X_A H_B^{(2)}, \quad (21)$$

where $H_A^{(2)}$ are given by eq.(18), and $\eta_{(3)}^{AB}$ is defined by

$$\eta_{(3)}^{AB} \eta_{BC}^{(3)} = \delta^A_C. \quad (22)$$

The triplet X transforms as $X \rightarrow U^{-1} X U$, where U is an element of $SU(1, 1)$ defined by eq.(14). We also obtain the following transformation law by taking the Hermitian conjugate,

$$\left(\eta^{(2)} X^\dagger \eta^{(2)} \right) \rightarrow U^{-1} \left(\eta^{(2)} X^\dagger \eta^{(2)} \right) U. \quad (23)$$

The transformation properties that we presented in this subsection would be sufficient for constructing invariants and covariant objects from doublets and triplets.

III. MODELS WITH GLOBAL $SU(1, 1)$ SYMMETRY

In this section, we present two examples of models with a noncompact *global* symmetry. We show that the models are free from ghosts if we impose appropriate conditions on the superpotential and Kähler potential. Note that the gauge kinetic function is not introduced in the case of a global symmetry.

If we give up the renormalizability in the traditional sense, we are left with no definite principle for determining the form of the superpotential, Kähler potential or the gauge kinetic function. In our construction of the models, we introduce particular type of functions, which might be unfamiliar. However, it is probably the simplest way to realize the properties that are necessary for the construction without making unreasonable assumptions. We will give an argument to support the introduction of inverse type Kähler potentials. Furthermore, it does not seem to pose any immediate problems as far as we understand.

Before presenting the models, we give a useful formula for a Kähler metric and study, with generality to some extent, the conditions for a Kähler potential that gives rise to a positive-definite Kähler metric. For an illustration of a possibly general property, we consider a Kähler potential that depends on a single triplet of $SU(1, 1)$ denoted by X_A , ($A = 1, 2, 3$) in such a way that its dependence is only through the variable $x \equiv \eta_{(3)}^{AB} X_A^\dagger X_B$, which is real and invariant under the group. The general form of the Kähler metric for this type of Kähler potential can be expressed as

$$K^{AB}(x) = \frac{\partial^2 K}{\partial X_A^\dagger \partial X_B} = K'(x) \eta_{(3)}^{AB} + K''(x) \left(\eta_{(3)}^{AC} \eta_{(3)}^{BD} X_C X_D^\dagger \right), \quad (24)$$

where

$$K'(x) \equiv \frac{dK}{dx}, \quad K''(x) \equiv \frac{d^2 K}{dx^2}. \quad (25)$$

It is easy to find that, due to the symmetry property of the group, the inverse of the Kähler metric is expressed as

$$(K^{-1})_{AB} = a(x) \eta_{AB}^{(3)} + b(x) X_A X_B^\dagger, \quad a(x) = \frac{1}{K'}, \quad b(x) = -\frac{K''}{K'(K' + xK'')}. \quad (26)$$

Now, we use the following property that holds for a general Hermitian three-by-three matrix to find the conditions to realize a positive-definite Kähler metric. Let M be an Hermitian three-by-three matrix. The eigenvalues of M are all real and positive if and only if it satisfies the following three conditions,

$$\det M > 0, \quad \text{tr} M > 0, \quad \text{tr} M^{-1} > 0. \quad (27)$$

By applying them to the case of the Kähler metric given in eq.(24), we obtain the following

conditions for a Kähler potential expressed as

$$0 < \det K^{AB} = K'^2 (K' + xK''), \quad (28)$$

$$0 < \text{tr} K^{AB} = -K' + K'' (|X_1|^2 + |X_2|^2 + |X_3|^2), \quad (29)$$

$$0 < \text{tr} (K^{-1})_{AB} = -\frac{1}{K'} - \frac{K''}{K' (K' + xK'')} (|X_1|^2 + |X_2|^2 + |X_3|^2), \quad (30)$$

which then lead to the following conditions

$$0 < K' + xK'', \quad (31)$$

$$\frac{K' + xK''}{2K''} < |X_3|^2, \quad 0 < K'', \quad (32)$$

$$-\frac{K'}{2K''} < |X_3|^2, \quad K' < 0, \quad (33)$$

correspondingly. The boundary in terms of x where at least one of the eigenvalues changes the sign is given by the condition $\det K^{AB} = 0, \pm\infty$. We will elaborate on the physical implications of the boundaries later. It is very important to note that these conditions are consistent with $SU(1,1)$ symmetry. This follows from two important properties; $SU(1,1)$ invariance and the Hermiticity of the matrix. Therefore, the positivity of the Kähler metric is a gauge-invariant concept.

Let us consider a Kähler potential that is a monomial of the form

$$K(x) = cx^n, \quad (34)$$

where c is a real constant and n is an integer. It is easy to confirm that the conditions given in eq.(31), (32), and(33) can be satisfied, namely the eigenvalues can become all positive, only for $x > 0$ and when the parameters are chosen to be $0 < c$ and $n < 0$. This means that the Kähler potential must be of the inverse type. We can not claim that the statement applies to a more general class of Kähler potentials from the argument that we presented here, but the similar argument may lead to such conclusion. At least, it applies to the case of the Kähler potential that will be introduced in the next subsection.

A. two doublet model

In this subsection, we present a model with the *global* $SU(1,1)$ symmetry. We introduce two chiral superfields Ψ and Ψ' that transform as the $SU(1,1)$ doublets. The superpotential

and the Kähler potential are given by

$$W(\Psi, \Psi') = M\phi + \frac{\lambda}{\phi}, \quad (35)$$

$$K(\Psi, \Psi^\dagger, \Psi', \Psi'^\dagger) = \frac{g_n}{(\Psi^\dagger \cdot \Psi)^n} + \frac{g'_n}{(\Psi'^\dagger \cdot \Psi')^n}, \quad (36)$$

where n is an integer to specify a model. The invariants are given as follows,

$$\phi \equiv 2(\Psi^T H_2 \Psi'), \quad \Psi^\dagger \cdot \Psi \equiv \Psi^\dagger \eta^{(2)} \Psi, \quad \Psi'^\dagger \cdot \Psi' \equiv \Psi'^\dagger \eta^{(2)} \Psi'. \quad (37)$$

Note that we need at least two doublets to write down a superpotential because of the symmetry property. The dimensional parameters are assumed to take the values expected from the assumption that we explained in subsection II A;

$$M = \tilde{M} M_{low}, \quad \lambda = \tilde{\lambda} M_{low}^5, \quad g_n = \tilde{g}_n M_{low}^{2n+2}, \quad g'_n = \tilde{g}'_n M_{low}^{2n+2}, \quad (38)$$

where dimensionless parameters $\tilde{M}, \tilde{\lambda}, \tilde{g}_n$ and \tilde{g}'_n are $O(1)$.

The scalar potential of the model has a contribution only from the F-term and can be obtained using the formula given in eq.(5). In order to calculate V_F , it is useful to introduce the components of the doublets in the following way

$$\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \Psi' \equiv \begin{pmatrix} \psi'_1 \\ \psi'_2 \end{pmatrix}. \quad (39)$$

The invariants are expressed in terms of these variables as

$$\phi = \psi_1 \psi'_2 - \psi_2 \psi'_1, \quad (40)$$

$$\Psi^\dagger \cdot \Psi = |\psi_1|^2 - |\psi_2|^2, \quad (41)$$

$$\Psi'^\dagger \cdot \Psi' = |\psi'_1|^2 - |\psi'_2|^2. \quad (42)$$

The Kähler metric for the model is expressed in terms of the components as

$$\begin{aligned} K^{\bar{i}j} &= \begin{pmatrix} K_{(1)} & 0 \\ 0 & K_{(2)} \end{pmatrix}, \\ K_{(1)} &= g_n n (\Psi^\dagger \cdot \Psi)^{-n-2} \begin{pmatrix} n|\psi_1|^2 + |\psi_2|^2 & -(n+1)\psi_1 \psi_2^* \\ -(n+1)\psi_1^* \psi_2 & |\psi_1|^2 + n|\psi_2|^2 \end{pmatrix}, \\ K_{(2)} &= g'_n n (\Psi'^\dagger \cdot \Psi')^{-n-2} \begin{pmatrix} n|\psi'_1|^2 + |\psi'_2|^2 & -(n+1)\psi'_1 \psi'^{\star}_2 \\ -(n+1)\psi'^{\star}_1 \psi'_2 & |\psi'_1|^2 + n|\psi'_2|^2 \end{pmatrix}, \end{aligned} \quad (43)$$

where we have assigned $\Phi_i = (\psi_1, \psi_2, \psi'_1, \psi'_2)$ to be used in the formula.

In order to determine the vacuum and the conditions for its existence, we use a simplified parametrization of the fields in our analysis. Before proceeding, we should make a few comments about this point. The choice of parametrization can have an important consequence, aside from the conventional matter, in the case of a noncompact group. It can be shown that, by exploiting $SU(1,1)$ symmetry, any configuration of Ψ can be brought to the form

$$\Psi = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad \Psi' = \begin{pmatrix} v_1 e^{i\beta_1} \\ v_2 e^{i\beta_2} \end{pmatrix}, \quad (44)$$

if Ψ satisfies the following condition

$$\Psi^\dagger \cdot \Psi > 0, \quad (45)$$

where u, v_1, v_2, β_1 , and β_2 are real. This condition implies $u \neq 0$. It is important to note that the condition given here is imposed in an $SU(1,1)$ invariant manner, and thus it is compatible with the symmetry. We should also point out that the configurations that do not obey eq.(45) are not related to the ones given in eq.(44) by any choice of parameters of $SU(1,1)$ transformations. It may be seen that the space of configurations in the case of noncompact group is divided into distinct sectors. See appendix A for details. Because there is no symmetry transformation left to simplify the parametrization any further, it is sufficient to deal with only five variables. If they acquire nonzero VEV's, it means that the symmetry is spontaneously broken. Throughout the analysis of two doublet models, we assume the condition for the VEV's given in eq.(45).

In the following, we study the Kähler metric and the scalar potential V_F to look for the consistent solutions and determine the conditions to realize what we need. First, we substitute the parametrization for the presumed vacuum given in eq.(44) into the eq.(43) and examine the eigenvalues. Assuming that the VEV's take nonzero values, we obtain the eigenvalues

$$g_n n u^{-2(n+1)}, \quad g_n n^2 u^{-2(n+1)},$$

$$\frac{1}{2} g'_n n (v_1^2 - v_2^2)^{-(n+2)} \left[(n+1)(v_1^2 + v_2^2) \pm \sqrt{(n+1)^2 (v_1^2 + v_2^2)^2 - 4n (v_1^2 - v_2^2)^2} \right]. \quad (46)$$

We notice the factor n and n^2 along with the rest of the common factors of the first two eigenvalues listed in eq.(46). It implies that n needs to be positive to realize the positive-

definite Kähler metric. For an arbitrary integer n , the term in the square root is positive for generic values of v_1 and v_2 . Thus, all the eigenvalues are guaranteed to be real. It can be shown that the terms in the square bracket of the third (with $+$ sign) and the fourth (with $-$ sign) eigenvalues in eq.(46) are positive for positive integer n . Furthermore, when n is a positive even integer and the coupling constants satisfy

$$g_n > 0, \quad g'_n > 0, \quad (47)$$

all the eigenvalues become positive as long as the VEV's are nonzero. Next, we consider the F-term contribution to the scalar potential V_F and examine the VEV's.

By substituting the parametrization given in eq.(44) and using the formula given in eq.(5), we obtain the expression for the scalar potential $V_F(u, v_1, v_2, \beta_1, \beta_2)$,

$$\begin{aligned} V_F &= M^2 \left\{ (K^{-1})_{1\bar{1}} v_2^2 + (K^{-1})_{2\bar{2}} v_1^2 + (K^{-1})_{4\bar{4}} u^2 \right\} \left| 1 - \frac{\Omega}{(uv_2 e^{i\beta_2})^2} \right|^2 \\ &= \frac{M^2 u^2 (nv_1^2 + v_2^2)}{g_n g'_n n^2} \left(g'_n u^{2n} + g_n (v_1^2 - v_2^2)^n \right) \left| 1 - \frac{\Omega}{(uv_2 e^{i\beta_2})^2} \right|^2, \quad \Omega \equiv \left(\frac{\lambda}{M} \right), \end{aligned} \quad (48)$$

where the quantity $(uv_2 e^{i\beta_2})$ is the VEV of the $SU(1,1)$ invariant ϕ defined in eq.(40). The explicit expression for V_F above shows that, when n is a positive even integer and the coupling constants satisfy eq.(47), it is bounded from below. Also, the eigenvalues of the Kähler metric given in eq.(46) are all positive automatically under these conditions.

The scalar potential of the present model has a remarkable feature, i.e. $V_F \geq 0$. This is a consequence of supersymmetry. It is interesting to note that this property is shared even with theories based on noncompact groups. However, supersymmetry alone is not sufficient to ensure the boundedness of the energy. In fact, the potential could become unbounded, for instance, when n is a positive odd integer. We conclude that the property $V_F \geq 0$ shows up only when the Kähler metric is positive-definite.

The next task is to find the vacuum and see whether the VEV's of the fields take nonzero values as we have assumed. We seek for the vacuum configurations that give $V_F = 0$. First of all, from the explicit expression for V_F , we see that the vacuum is realized only for $uv_2 \neq 0$. The configurations of the vacua are obtained by solving the following equation

$$\left| 1 - \frac{\Omega}{(uv_2 e^{i\beta_2})^2} \right| = 0. \quad (49)$$

We find the solutions to the equation for both $\Omega > 0$ and $\Omega < 0$ respectively as

$$uv_2 e^{i\beta_2} = |\Omega|^{\frac{1}{2}} e^{i\pi l_+}, \quad l_+ = 0, 1 \quad (\Omega > 0), \quad (50)$$

$$uv_2 e^{i\beta_2} = |\Omega|^{\frac{1}{2}} e^{i\frac{\pi}{2} + i\pi l_-}, \quad l_- = 0, 1 \quad (\Omega < 0), \quad (51)$$

which lead to the relations for the VEV's as follows

$$uv_2 = \pm |\Omega|^{\frac{1}{2}}, \quad \sin \beta_2 = 0 \quad (\Omega > 0), \quad (52)$$

$$uv_2 = \pm |\Omega|^{\frac{1}{2}}, \quad \cos \beta_2 = 0 \quad (\Omega < 0). \quad (53)$$

In either case, only the combination uv_2 is fixed by the stationary condition. Note that we have $uv_2 = O(M_{low}^2)$. The VEV's v_1 and β_1 are allowed to take arbitrary values while giving $V_F = 0$. Thus, we have degenerate vacua or moduli parametrized by these variables for fixed values of the parameters of the model. Note that each point in the space of degenerate vacua corresponds to the inequivalent physics.

Now, we have shown that the following is achieved for the present model by choosing n to be positive even integer and the coupling constants g_n and g'_n to satisfy eq.(47): (i) Stable and degenerate vacua exist. (ii) All the vacua break $SU(1,1)$ symmetry by having nonzero VEV's of the fields. (iii) The Kähler metric at the vacuum is positive-definite. This guarantees that no ghost appears in the fermion fields as well. Note that these three properties are realized simultaneously in a consistent manner. As a result, the model is free from ghosts as we claimed at the beginning of the section. We remark that there is no unbroken symmetry left at the vacuum of the two doublet model. We also note that the vacua do not break supersymmetry.

B. one triplet model

In this subsection, we present another model that has the *global* $SU(1,1)$ symmetry. We introduce a single chiral superfield X that transforms as an adjoint representation of the group, i.e. an $SU(1,1)$ triplet. The superpotential and Kähler potential of the model are given by

$$W(X) = M(X \cdot X) + \frac{\lambda_m}{(X \cdot X)^m}, \quad (54)$$

$$K(X, X^\dagger) = \frac{g_n}{(X^\dagger \cdot X)^n}, \quad (55)$$

where m and n are integers to specify a model. The dimensional parameters M , λ_m , and g_n are assumed to take the values that follow from the assumption explained in section II A. The invariants are given as follows,

$$(X \cdot X) \equiv 2\text{tr}(X^2) = \eta_{(3)}^{AB} X_A X_B, \quad (56)$$

$$(X^\dagger \cdot X) \equiv 2\text{tr}(\eta^{(2)} X^\dagger \eta^{(2)} X) = \eta_{(3)}^{AB} X_A^\dagger X_B, \quad (57)$$

where the components of X are defined in eq.(21).

In the analysis of one triplet model, we make the choice of the parametrization much the same way as we did in the two doublet model. It can be shown that, by exploiting the $SU(1, 1)$ symmetry, any configuration of X can be brought to the following form

$$X_A = (0, v_2, v_3 e^{i\alpha}), \quad (58)$$

where v_2, v_3 and α are real, if it satisfies a certain condition (see appendix A). In order to state the condition clearly, we express X_A in terms of real and imaginary parts as $X_A = Y_A + iZ_A$, where Y_A and Z_A are real. The condition on the parametrization of the field is expressed as

$$(Z \cdot Z) > 0, \quad (59)$$

which implies

$$v_3 \neq 0, \quad \sin \alpha \neq 0. \quad (60)$$

No condition is imposed on the real part of X_A . Throughout the following analysis, we assume the condition given in eq.(60).

The Kähler metric for the model is obtained as

$$K^{AB} = ng_n (X^\dagger \cdot X)^{-n-2} \times \begin{pmatrix} n|X_1|^2 - |X_2|^2 + |X_3|^2 & (n+1)X_1 X_2^* & -(n+1)X_1 X_3^* \\ (n+1)X_1^* X_2 & -|X_1|^2 + n|X_2|^2 + |X_3|^2 & -(n+1)X_2 X_3^* \\ -(n+1)X_1^* X_3 & -(n+1)X_2^* X_3 & |X_1|^2 + |X_2|^2 + n|X_3|^2 \end{pmatrix}, \quad (61)$$

where we have assigned the components as $\Phi_i = (X_1, X_2, X_3)$ to be used in the formula given in eq.(8). In order to see whether the Kähler metric at the vacuum can be positive-definite,

we substitute the parametrization given in eq.(58). Assuming that v_2, v_3 , and α take nonzero VEV's, the eigenvalues of the Kähler metric evaluated at these presumed VEV's are found to be

$$\begin{aligned} & ng_n (-v_2^2 + v_3^2)^{-n-1}, \\ & \frac{ng_n}{2} (-v_2^2 + v_3^2)^{-n-2} \\ & \times \left[(n+1) (v_2^2 + v_3^2) \pm \sqrt{(n+1)^2 (v_2^2 + v_3^2)^2 - 4n (-v_2^2 + v_3^2)^2} \right]. \end{aligned} \quad (62)$$

The term in the square root is positive for generic values of v_2, v_3 and α for an arbitrary integer n , and thus the eigenvalues are real. It can also be shown that the terms in the square bracket of the second(+ sign) and third(- sign) eigenvalues are positive for a positive integer n . However, any choice of the parameters of the model does not guarantee the positive-definiteness of the Kähler metric for generic v_2 and v_3 . From the general argument given at the beginning of this section, we know that the parameter of the model must be chosen to be $g_n > 0$ and we need the VEV to satisfy $(-v_2^2 + v_3^2) > 0$. For this purpose, we need to examine the actual VEV's to see if such a situation is realized.

Following the formula given in eq.(5) and substituting the parametrization given in eq.(58), we obtain the expression for the scalar potential $V_F(v_2, v_3, \alpha)$. In doing this, the formula for the inverse of the Kähler metric given at the beginning of this section might be useful. The expression for the potential is given as

$$V_F = \frac{4M^2}{n^2 g_n} (-v_2^2 + v_3^2)^{n+2} \left| 1 - \frac{\Omega_m}{(-v_2^2 + v_3^2 e^{i2\alpha})^{m+1}} \right|^2, \quad \Omega_m \equiv \left(\frac{m\lambda_m}{M} \right). \quad (63)$$

It is easy to see that, when n is a positive even integer and the coupling constant satisfies

$$g_n > 0, \quad (64)$$

we have $V_F \geq 0$. Again, this is a consequence of supersymmetry. Unfortunately, in contrast to the case of the two doublet model, the conditions that lead to the bounded potential do not guarantee the positive-definiteness of the Kähler metric for generic values of v_2 and v_3 . From the explicit expression for the eigenvalues of the Kähler metric, we see that the VEV of the field must satisfy the relation

$$(X^\dagger \cdot X) \Big|_{VEV} = -v_2^2 + v_3^2 > 0, \quad (65)$$

in order for the metric to be positive-definite. Note that the quantity is the VEV of the $SU(1, 1)$ invariant. We need to examine the actual VEV's to find whether the condition is satisfied at the vacuum.

To do this, we look for solutions to the following equation,

$$\left| 1 - \frac{\Omega_m}{(-v_2^2 + v_3^2 e^{i2\alpha})^{m+1}} \right| = 0. \quad (66)$$

The solution for $\Omega_m > 0$ and $\Omega_m < 0$ are given as

$$(-v_2^2 + v_3^2 e^{i2\alpha}) = |\Omega_m|^{\frac{1}{m+1}} e^{i\frac{2\pi}{m+1}l_+}, \quad l_+ = 0, 1, \dots, m \quad (\Omega_m > 0), \quad (67)$$

$$(-v_2^2 + v_3^2 e^{i2\alpha}) = |\Omega_m|^{\frac{1}{m+1}} e^{\frac{i\pi}{m+1}(1+2l_-)}, \quad l_- = 0, 1, \dots, m \quad (\Omega_m < 0), \quad (68)$$

where the term on the left-hand side is the VEV of $SU(1, 1)$ invariant $(X \cdot X)$. Note that, in order for a solution of the equation to be a configuration that gives $V_F = 0$, it must satisfy

$$(X \cdot X) \Big|_{VEV} = (-v_2^2 + v_3^2 e^{i2\alpha}) \neq 0. \quad (69)$$

It is important to note that, because of the condition given in eq.(60), we need to see if the solutions are acceptable as vacuum configurations. In the following, we will be interested in the case of the superpotential with $m = 2$.

Firstly, for $\Omega_2 > 0$, we naively have three types of solutions, which are specified by the labels $l_+ = 0, 1, 2$ in eq.(67). However, we find that the one with $l_+ = 0$ is not acceptable because of the condition given in eq.(60). The rest of the solutions, i.e. those with $l_+ = 1, 2$, is eligible for the vacuum and leads to the following relations for the VEV's

$$\begin{aligned} v_3^2 &= \sqrt{\left(v_2^2 - \frac{1}{2}|\Omega_2|^{\frac{1}{3}}\right)^2 + \frac{3}{4}|\Omega_2|^{\frac{2}{3}}}, \quad \sin(2\alpha) = \frac{\pm \frac{\sqrt{3}}{2}|\Omega_2|^{\frac{1}{3}}}{\sqrt{\left(v_2^2 - \frac{1}{2}|\Omega_2|^{\frac{1}{3}}\right)^2 + \frac{3}{4}|\Omega_2|^{\frac{2}{3}}}}, \\ \cos(2\alpha) &= \frac{v_2^2 - \frac{1}{2}|\Omega_2|^{\frac{1}{3}}}{\sqrt{\left(v_2^2 - \frac{1}{2}|\Omega_2|^{\frac{1}{3}}\right)^2 + \frac{3}{4}|\Omega_2|^{\frac{2}{3}}}} \quad (\text{type F1}), \end{aligned} \quad (70)$$

which express v_3 and α in terms of v_2 , where v_2 is allowed to take arbitrary values. We refer to this type of vacua as “type F1”. The model possesses degenerate vacua parametrized by a continuous variable v_2 . Note that we have $v_3 = O(M_{low})$ if we choose $v_2 = O(M_{low})$. Let us look at the eigenvalues of Kähler metric at these vacua. In order to examine whether the condition given in eq.(65) is satisfied, we evaluate the following quantity

$$v_3^4 - v_2^4 = |\Omega_2|^{\frac{1}{3}} \left(|\Omega_2|^{\frac{1}{3}} - v_2^2 \right). \quad (71)$$

Then, it is easy to see that the Kähler metric is positive-definite in the subregion of type F1 vacua that is specified by

$$0 \leq v_2^2 < |\Omega_2|^{\frac{1}{3}}, \quad (72)$$

and thus the ghost is absent in this region of the vacua. On the other hand, in the rest of the region of the type F1 vacua, the Kähler metric is not positive-definite and the ghost does appear in the theory there.

We have found that the type F1 vacua are separated into two distinct regions by points specified by $v_2^2 = |\Omega_2|^{\frac{1}{3}}$ in eq.(70); one with a ghost and the other without a ghost. We call the former a “ghost” phase and the latter a “ghost free” phase. Because the potential energy is the same $V_F = 0$ at all points within the type F1 vacua, one might worry about the possibility of a transition of VEV’s between the two phases. If such a transition is allowed, the theory would be ill-defined. It turns out that this does not likely to happen. We will give the argument for this below.

To explain the basic idea of the argument about the transition in our model, let us consider a scalar field ϕ with a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}K(\phi)(\partial_\mu\phi)^2 - V(\phi), \quad V(\phi) = K^{-1}(\phi)f(\phi), \quad (73)$$

and the corresponding Hamiltonian given as

$$\mathcal{H} = \frac{1}{2}K(\phi) \left[\dot{\phi}^2 + (\nabla\phi)^2 \right] + V(\phi). \quad (74)$$

Note that the kinetic and potential terms are related to each other through a single function $K(\phi)$ just as in our supersymmetric models. We assume that the potential has degenerate minima at $V = 0$. We also assume that the function K do not have zeros in the space of vacua, which is similar to the case of our one triplet model, but we may have points or regions characterized by the VEV v_{crit} that gives $K^{-1}(v_{crit}) = 0$ within the vacua. Suppose that a transition from one point in the space of vacua specified by the VEV v_i of ϕ to another one specified by v_f occurs. Then, it must involve a configuration ϕ_{fi} that connects these two VEV’s. Transitions within the ghost free region occurs with a configuration that gives finite Hamiltonian density. If we consider a transition from one point in the ghost free phase to another one in the ghost phase, the corresponding configurations necessarily involve the critical point v_{crit} at which we have $K^{-1} = 0$, and thus formally $\mathcal{H} \rightarrow +\infty$. This

suggest that the transitions that cross critical points do not take place with finite energy fluctuations. As the VEV's approach the critical value, the kinetic term dominates over the potential term and thus, it approaches a free theory.

Let us look at the case of $\Omega_2 < 0$. Naively, we have three types of solutions; the ones with $l_- = 0, 1, 2$ in eq.(68). Actually, all three of them correspond to the vacuum configuration, and are classified into the following two types of relations for the VEV's. Those that are derived from $l_- = 1$ are

$$v_3^2 = |\Omega_2|^{\frac{1}{3}} - v_2^2, \quad \cos(2\alpha) = -1 \quad (\text{type F2}), \quad (75)$$

which are valid in the region of v_2 specified by $0 \leq v_2^2 \leq |\Omega_2|^{\frac{1}{3}}$. We refer to this as “type F2” vacua. Note that we have $v_3 = O(M_{low})$ in this region. We see that in the region specified by

$$0 \leq v_2^2 < \frac{1}{2}|\Omega_2|^{\frac{1}{3}}, \quad (76)$$

the positivity condition given in eq.(65) is satisfied. The Kähler metric is positive-definite and thus the ghost is absent there. In contrast, in the region $\frac{1}{2}|\Omega_2|^{\frac{1}{3}} < v_2^2 \leq |\Omega_2|^{\frac{1}{3}}$ the Kähler metric is not positive-definite and thus the ghost appears. The relations for the VEV's that are derived from $l_- = 0, 2$ given in eq.(68) are

$$\begin{aligned} v_3^2 &= \sqrt{\left(v_2^2 + \frac{1}{2}|\Omega_2|^{\frac{1}{3}}\right)^2 + \frac{3}{4}|\Omega_2|^{\frac{2}{3}}}, \quad \sin(2\alpha) = \frac{\pm \frac{\sqrt{3}}{2}|\Omega_2|^{\frac{1}{3}}}{\sqrt{\left(v_2^2 + \frac{1}{2}|\Omega_2|^{\frac{1}{3}}\right)^2 + \frac{3}{4}|\Omega_2|^{\frac{2}{3}}}}, \\ \cos(2\alpha) &= \frac{v_2^2 + \frac{1}{2}|\Omega_2|^{\frac{1}{3}}}{\sqrt{\left(v_2^2 + \frac{1}{2}|\Omega_2|^{\frac{1}{3}}\right)^2 + \frac{3}{4}|\Omega_2|^{\frac{2}{3}}}} \quad (\text{type F3}), \end{aligned} \quad (77)$$

which are also expressed in terms of v_2 . Note that they are valid for arbitrary values of v_2 . We refer to this as “type F3” vacua. For the type F3 VEV's, the following property is satisfied for an arbitrary value of v_2

$$v_3^4 - v_2^4 = |\Omega_2|^{\frac{1}{3}} \left(|\Omega_2|^{\frac{1}{3}} + v_2^2 \right) > 0. \quad (78)$$

Consequently, the ghosts are absent at every point in the space of type F3 vacua. We see that $v_3 = O(M_{low})$ if we choose $v_2 = O(M_{low})$. For the same reason that we explained about the transition between regions with and without ghosts, a theory with $\Omega_2 < 0$ in the ghost free phase do not cross the critical region and thus it is well-defined.

Our analysis here has demonstrated that the following is achieved by choosing n to be positive even integer and the coupling constant g_n to satisfy eq.(64): (i) Stable and degenerate vacua exist for both $\Omega_2 > 0$ and $\Omega_2 < 0$ cases. The VEV v_2 is not fixed by the requirement of the minimum energy condition. We should point out that the vacuum configuration specified by $v_2 = 0$ is a part of vacua of the present model. (ii) All the vacua break $SU(1,1)$ symmetry. For the vacuum with $v_2 \neq 0$, $SU(1,1)$ is completely broken and no unbroken symmetry is left. At the vacua with $v_2 = 0$, the $U(1)$ symmetry, which is the maximal compact subgroup of $SU(1,1)$, is left unbroken. (iii) The Kähler metric is positive-definite in the certain subregion of the vacua, which we refer to as ghost free phase. The positivity of the Kähler metric guarantees that no ghost appears in the fermion fields. The transition between the ghost free phase and the ghost phase are highly suppressed. These three properties are realized simultaneously in a consistent manner. As a result, the one triplet model with global $SU(1,1)$ symmetry can be defined without the appearance of ghosts just as we claimed at the beginning of the section. We also note that supersymmetry is unbroken at the vacuum.

Although we have not presented the results for the case of superpotential with $m = 1$ in this paper, one can confirm that stable vacua and the positive-definite Kähler metric are obtained. However, when the symmetry is made local, no solution exists that realizes $V_F = 0$, $V_D = 0$ and the positive-definite metric simultaneously. This does not immediately lead to the conclusion that theory is ill-defined, because the possibility of a vacuum with $V \neq 0$ is not excluded. However, the analyses become more involved due to the lack of manifest vacuum, i.e. $V = 0$. Because our purpose of the present paper is to present an example of a viable model, we consider the one that allows a simple analysis.

IV. MODELS WITH LOCAL $SU(1,1)$ SYMMETRY

In this section, we consider the gauging of the $SU(1,1)$. In the case of a local symmetry, it is the sum of the two $V = V_F + V_D$ that we need to look at in order to examine the stability of the system. Since we already have the expression for V_F in each model, what we need to calculate is V_D . We show that one triplet model become free of ghosts when certain conditions on the parameters of the superpotential, Kähler potential, and the gauge kinetic function are satisfied.

Before proceeding, we give a useful formula for the inverse of the gauge kinetic function f_{AB} to be used in the calculation of V_D . Suppose that f_{AB} is a function of fields, collectively denoted by Φ of the following form

$$f_{AB}(\Phi) = -\eta_{AB}^{(3)} + w(\Phi)N_A N_B, \quad (79)$$

where N_A is a function of Φ that transforms as an adjoint representation of the group and $w(\Phi)$ is an invariant function of Φ . Note that N_A is an arbitrary function of Φ , which is not restricted to be quadratic in Φ . Then, its inverse $(f^{-1})^{AB}$ is given as

$$(f^{-1})^{AB}(\Phi) = -\eta_{(3)}^{AB} - \frac{w}{1 - w(N \cdot N)} N^A N^B. \quad (80)$$

Unfortunately, it seems difficult to state the conditions that ensure the positivity of the real part of f_{AB} , unlike the case of the Kähler metric due to the non-Hermiticity of f_{AB} .

A. two doublet model

In this subsection, we consider the two doublet model, which we have studied in subsection III A, but with the local $SU(1, 1)$ symmetry. The model is described by the same superpotential and the Kähler potential given in eq.(36). The gauge kinetic function for the model is given by

$$f_{AB}(\Psi, \Psi') = -\eta_{AB}^{(3)} + \xi \frac{(\Psi\Psi')_A (\Psi\Psi')_B}{(\Psi\Psi\Psi'\Psi')_{singlet}}, \quad (81)$$

where ξ is a dimensionless parameter of the model and the indices A and B run over the adjoint representation of $SU(1, 1)$. For simplicity, we confine our investigation to the case of real ξ . The covariant and invariant objects are introduced as

$$\begin{aligned} (\Psi\Psi')_A &\equiv 4 \left(\Psi^T H_2^{(2)} H_A^{(2)} \Psi' \right), \\ &= (i(\psi_1\psi'_1 - \psi_2\psi'_2), -(\psi_1\psi'_1 + \psi_2\psi'_2), -(\psi_1\psi'_2 + \psi_2\psi'_1)) \end{aligned} \quad (82)$$

$$(\Psi\Psi\Psi'\Psi')_{singlet} = \phi^2, \quad (83)$$

where ϕ is defined in eq.(37) and the components of Ψ and Ψ' are introduced as in eq.(39). One might notice that the reason for introducing two doublets in the present model is to form an invariant that is quartic in fields and to keep ξ a dimensionless parameter.

Before examining whether the real part of gauge kinetic function evaluated at the vacuum is positive-definite, we substitute the parametrization given in eq.(44) and study their properties for generic values of v_1, v_2, β_1 and β_2 . We obtain for $f_{AB}(v_1, v_2, \beta_1, \beta_2)$

$$f_{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{\xi}{v_2^2} \begin{pmatrix} -v_1^2 e^{i2(\beta_1-\beta_2)} & -iv_1^2 e^{i2(\beta_1-\beta_2)} & -iv_1 v_2 e^{i(\beta_1-\beta_2)} \\ -iv_1^2 e^{i2(\beta_1-\beta_2)} & v_1^2 e^{i2(\beta_1-\beta_2)} & v_1 v_2 e^{i(\beta_1-\beta_2)} \\ -iv_1 v_2 e^{i(\beta_1-\beta_2)} & v_1 v_2 e^{i(\beta_1-\beta_2)} & v_2^2 \end{pmatrix}. \quad (84)$$

The eigenvalues of the real part of f_{AB} are given as follows

$$1 - \xi \left(\frac{v_1}{v_2} \right)^2, \quad \frac{1}{2v_2^2} \left[(v_1^2 + v_2^2) \xi \pm \sqrt{(\xi - 2)^2 v_2^4 + \xi^2 v_1^4 + 2v_1^2 v_2^2 \xi (\xi + 2)} \right], \quad (85)$$

for $2(\beta_1 - \beta_2) = l\pi$, where l is an integer. It can be shown that, for generic values of v_1 and v_2 , any choice of ξ does not guarantee the positivity of the eigenvalues. Note that we give the explicit form of the eigenvalues only for $2(\beta_1 - \beta_2) = l\pi$, but the results are basically the same for other values. We need to find the minimum of $V = V_F + V_D$ and examine the VEV's of the fields to see whether the positive-definite metric can be realized.

By substituting the parametrization given in eq.(44) into the formula given in eq.(7) and after somewhat lengthy calculations, we obtain for $V_D(u, v_1, v_2, \beta_1, \beta_2)$

$$V_D = \frac{g^2}{8r} \left[\left\{ 2(-rv_1^2 + v_2^2) \left(1 + ng'_n [v_1^2 - v_2^2]^{-(n+1)} \right) + u^2 (1 + ng_n u^{-2(n+1)}) \right\}^2 - 4rv_1^2 (v_1^2 \xi \cos(2\beta) - v_2^2) \left(1 + ng'_n [v_1^2 - v_2^2]^{-(n+1)} \right)^2 \right], \quad (86)$$

where $r \equiv \xi - 1$, $\beta \equiv \beta_1 - \beta_2$, with the help of the formula for the inverse of f_{AB} given at the beginning of this section.

The term on the second line of eq.(86) does not have a definite sign irrespective of the choice of n and of the coupling constants of the model g_n, g'_n and ξ . It is a manifestation of a general property of theories based on a noncompact group that there appear directions in the field space that decrease the potential energy arbitrarily. In order for the potential to be bounded from below, those directions must disappear somehow. However, because such direction can not be eliminated by any choice of the parameters for the present case, the D-term potential V_D is not bounded from below. This provides another example that illustrates the fact that supersymmetry alone does not guarantee the boundedness of the potential. The two doublet model with local $SU(1, 1)$ symmetry does not possess a ground state and hence it can not be defined consistently.

B. one triplet model

In this subsection, we consider the model with one triplet, which we studied in subsection IIIB, but with the local $SU(1,1)$ symmetry. The superpotential and Kähler potential of the model are given by eqs.(54) and (55). The gauge kinetic function for the model is

$$f_{AB}(X) = -\eta_{AB}^{(3)} + \xi \frac{X_A X_B}{(X \cdot X)}, \quad (87)$$

where ξ is a dimensionless parameter and the indices A and B run over the adjoint representation of $SU(1,1)$. For simplicity, we confine ourself to the case of real ξ .

Just as in the case of the two doublet model, we first study the properties of the gauge kinetic function for generic VEV's of the fields and examine the positivity. To do this, we substitute the parametrization given in eq.(58) into eq.(87) and obtain for $f_{AB}(v_1, v_2, v_3, \alpha)$

$$f_{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + R \begin{pmatrix} 0 & 0 & 0 \\ 0 & v_2^2 & v_2 v_3 e^{i\alpha} \\ 0 & v_2 v_3 e^{i\alpha} & v_3^2 e^{i2\alpha} \end{pmatrix}, \quad R \equiv \frac{\xi}{(-v_2^2 + v_3^2)}. \quad (88)$$

The eigenvalues of the real part of f_{AB} are expressed as

$$1, \quad \frac{1}{2} \left[R (v_2^2 + v_3^2 \cos(2\alpha)) \pm \sqrt{(2 - R v_3^2 \cos(2\alpha))^2 + R^2 v_2^2 (v_2^2 + 2v_3^2) + 4R v_2^2} \right]. \quad (89)$$

Again, it is not possible to guarantee the positivity of the eigenvalues for generic values of v_2, v_3 and α by any choice of ξ . We need to look into the actual VEV's to find out whether it can be realized.

Let us find the minimum of the total potential $V = V_F + V_D$ and examine the VEV's of the fields. We already have the expression for V_F and the configurations that are determined by $V_F = 0$ alone. What we need to do is to obtain the explicit expression for V_D . It is carried out by substituting the parametrization given in eq.(58) into the formula given in eq.(7), and after somewhat lengthy calculations, we obtain the expression for $V_D(v_2, v_3, \alpha)$ as

$$V_D = \frac{2n^2 g^2 g_n^2}{(-v_2^2 + v_3^2)^{2(n+1)}} v_2^2 v_3^2 \sin^2 \alpha, \quad (90)$$

where we used the following expressions for the generators of $SU(1,1)$ in the adjoint representation $H_A^{(3)}$,

$$H_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +i \\ 0 & +i & 0 \end{pmatrix}, \quad H_2^{(3)} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad H_3^{(3)} = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (91)$$

Note that V_D does not depend on ξ . We immediately see that, when n is a positive integer and $g_n > 0$, V_D is bounded from below for generic values of v_2, v_3 and α . Therefore, the conditions that give rise to the bounded V_F (see the conditions just below eq.(63)) also ensure that V_D is bounded from below. To be more precise, we have $V_D \geq 0$, which implies $V \geq 0$. Thus, the minimum of the total potential could be realized as $V_F = V_D = 0$, which leads to the supersymmetric vacuum. In the following analysis, we choose the parameters of the model that lead to bounded V_F , i.e. n is a positive even integer and $g_n > 0$.

Taking into account the condition given in eq.(60), we find that there is only one type of configurations that realizes the minimum of V_D , that is $v_2 = 0$. This means that the subgroup $U(1)$ is left unbroken at the vacuum. In contrast to the requirement $V_F = 0$, the VEV's v_3 and α are not fixed by $V_D = 0$ and are allowed to be arbitrary. In the following, we present the analysis of the VEV's for both $\Omega_2 > 0$ and $\Omega_2 < 0$ cases combining the conditions from $V_F = 0$ and $V_D = 0$ including the evaluation of the eigenvalues of f_{AB} at each vacuum.

For the superpotential with $\Omega_2 > 0$, there is only one type of vacuum configurations. It is given by type F1 that we examined in subsection III B with $v_2 = 0$. We see that the space of vacua parametrized by v_2 in the case of the global symmetry shrinks to points specified by $v_2 = 0$ and certain discrete values of α in the case of the local symmetry. We refer to this as “type D1” vacua, which is specified by the following relations

$$v_2 = 0, \quad v_3^2 = |\Omega_2|^{\frac{1}{3}}, \quad \cos(2\alpha) = -\frac{1}{2} \quad (\text{type D1}). \quad (92)$$

We have already shown that the type D1 vacua give rise to the positive-definite Kähler metric (see eq.(72)). The eigenvalues of f_{AB} given in eq.(89) at these vacua are given as $\{1, 1, \frac{1}{2}(-2 - \xi)\}$. Obviously, with the choice of the parameter $\xi < -2$, all the eigenvalues become positive at all points in the type D1 vacua and thus the ghost is absent from the theory.

For the superpotential with $\Omega_2 < 0$, there are two types of degenerate vacua. One is of type F2 with the condition $v_2 = 0$, which is specified by the following

$$v_2 = 0, \quad v_3^2 = |\Omega_2|^{\frac{1}{3}}, \quad \cos(2\alpha) = -1 \quad (\text{type D2}). \quad (93)$$

We refer to this as “type D2” vacua. The eigenvalues of f_{AB} at these vacua are $\{1, 1, (-1 - \xi)\}$, which become all positive for the choice of the parameter $\xi < -1$. The

other is of type F3 with the condition $v_2 = 0$ and expressed as

$$v_2 = 0, \quad v_3^2 = |\Omega_2|^{\frac{1}{3}}, \quad \cos(2\alpha) = \frac{1}{2} \quad (\text{type D3}). \quad (94)$$

We refer to these as “type D3”. The eigenvalues of f_{AB} at these vacua are $\{1, 1, \frac{1}{2}(\xi - 2)\}$, which become all positive with the choice of the parameter $2 < \xi$. For the case $\Omega_2 < 0$, any choice of the parameter ξ can not realize the situation that there is no ghost phase. There are two phases in the space of vacua. The transition is expected to be highly suppressed for the same reason that we explained in subsection III B, and therefore the theory in the ghost free phase remains so.

We have shown that metrics for the kinetic terms for all the perturbative quanta are positive-definite if the appropriate parameters are chosen for the model. As another important point, we need to make sure that there are no problems with the mass terms for the gauge bosons. We find that the mass terms for the gauge bosons are the same for all types of vacua and they are given as

$$\mathcal{L}_{mass}^{gauge} = ng^2g_n |\Omega_2|^{-\frac{n}{3}} \left[(A_\mu^1)^2 + (A_\mu^2)^2 \right]. \quad (95)$$

Clearly, they have the correct signs at the vacua, which are necessary for consistency of the theory. We see that the masses of the gauge bosons are $O(M_{low})$. The absence of the mass term for A_μ^3 , which corresponds to gauge field of the compact part of the group, indicates that the gauge boson remains massless as expected from the property of the vacua of the one triplet model.

We would like to make a comment about the form of D-term contribution to the scalar potential. In particular, we point out that an important element that leads to bounded V_D is the availability of the symmetry transformations that take generic parametrization of the field into the one given in eq.(58). To see this, let us consider the most general parametrization of the field

$$X_A = (v_1 e^{i\alpha_1}, v_2 e^{i\alpha_2}, v_3 e^{i\alpha_3}), \quad (96)$$

where v_i and α_i , ($i = 1, 2, 3$) are real. The expression for $V_D(v_1, v_2, v_3, \alpha_1, \alpha_2, \alpha_3)$ is given by

$$V_D = \frac{2n^2 g^2 g_n^2}{(-v_1^2 - v_2^2 + v_3^2)^{2(n+1)}} \times \left[v_1^2 v_3^2 \sin^2(\alpha_1 - \alpha_3) + v_2^2 v_3^2 \sin^2(\alpha_2 - \alpha_3) - v_1^2 v_2^2 \sin^2(\alpha_1 - \alpha_2) \right]. \quad (97)$$

Apparently, the third term has a negative sign, which might lead to the unbounded potential. However, due to the existence of a symmetry transformation that allows us to set $\alpha_1 = \alpha_2$, it actually does not lead to instability. Of course, we can reach this conclusion by careful inspection of the explicit expression for V_D . Note that the symmetry transformation that we mention here is a part of those that allow the parametrization given in eq.(58).

Our analysis here has demonstrated that the following is achieved for the model by choosing n to be positive even integer and the coupling constant g_n to satisfy eq.(64): (i) Stable and degenerate vacua exist. (ii) All the vacua break $SU(1, 1)$ symmetry. However, the $U(1)$ symmetry, which is a maximal subgroup of $SU(1, 1)$ is left unbroken. (iii) The Kähler metric at each vacuum is positive-definite. (iv) The real part of gauge kinetic function at each vacuum is positive-definite. Because of these properties, no ghosts appear in the fermionic sector as well as in the bosonic sector. The transition from the ghost free phase to the ghost phase is not expected to occur. These three properties are realized simultaneously in a consistent manner. As a result, the one triplet models(a class of models specified by an integer n) are expected to be free from ghosts as we claimed at the beginning of the section. We also note that supersymmetry is not broken at each vacuum.

V. SUMMARY AND DISCUSSIONS

We have presented a construction of theories with a linearly realized $SU(1, 1)$ symmetry, which is the simplest noncompact nonabelian group, in the framework of $\mathcal{N} = 1$ global supersymmetry. In our construction, the symmetry is realized linearly without introducing the constrained fields. We first explained the problem of ghosts and discussed what must be achieved in order to solve it. For illustration of important points, we have presented two models both in global and local symmetry cases. In the global symmetry case, we have shown that both the two doublet and one triplet models satisfy all the requirements for the consistency; The symmetry is spontaneously broken, positive-definite Kähler metric is realized, and the energy is bounded from below. In the local symmetry case, two doublet model is shown to be inconsistent due to the lack of bounded potential. On the other hand, we have shown that the one triplet model satisfies all the requirements for consistency. Thus our analysis provides a suggestive evidence that it is possible to define a gauge theory based on a noncompact group. However, there is an important point that we must emphasize, i.e.

the assumptions on the dimensional coupling constants, which we explained in section II A, are the crucial elements for our construction. Hence the results of our analysis heavily depends on them. If these assumptions are shown to be inconsistent, our theory would be able to describe only the massless particles because all the massive particles would have masses of order of the cutoff M_{high} . Even in that case, our construction is still useful for description of the dynamics of the massless particles while realizing the symmetry linearly on the fields.

Although we have constructed models in the framework of $\mathcal{N} = 1$ supersymmetry, the supersymmetry may not be a necessity for constructing a ghost free theory with noncompact group. Rather, the incorporation of supersymmetry itself makes it difficult due to the severe restriction it imposes. For example, the potential term and the kinetic term are related to each other through a single function. However, this property plays an extremely important role in preventing the potential disaster from occurring. Due to the existence of ghost phase and ghost free phase, which have the same potential energy, there is a possibility of transition between them. In the supersymmetric framework, however, if we manage to construct a consistent model at the classical level, the phase transition is highly suppressed because it must involve a configuration that requires an infinite amount of energy. It is due to the linkage between the kinetic term and the potential term that the transitions are suppressed.

Perhaps, we should mention a few words about the inverse type potentials that we introduced in our construction. Because of their peculiar form, which has not been studied in the literature, one might even suspect that such potentials are not allowed in quantum field theories, e.g. they might violate locality or causality. At present, we do not have any theoretical argument to show that this is not the case. However, it is interesting to know that there are examples of such type of potentials that are generated by instanton effects in supersymmetric QCD [31, 32] though in different context.

We should also mention that our motivation for adopting the inverse type functions for the superpotential and that for the Kähler potential and gauge kinetic function are essentially different. The reason for the former is as follows. It has been chosen so that no vacua preserving the noncompact symmetry are allowed in the theory. To realize this, the inverse type potential is probably the simplest choice. However, we suspect that such condition is not necessary. What is actually needed is the existence of the symmetry breaking vacuum in

the theory. The coexistence of the broken and unbroken vacua does not cause any problem because transitions between them are suppressed. The reason for the latter is that this type of function is probably the only choice to realize the positive-definite metric without making unreasonable assumptions. If not for this type of functions, we might have to assume that the terms that would have been suppressed by large mass scales M_{high} make sizable contributions in order to reverse the sign of the metric. Note also that we have explicitly shown with certain generality that the inverse type function is necessary in the case of one triplet model in section III. The construction of nonsupersymmetric theories also must involve the inverse type functions in the kinetic terms as well.

In order to clarify our perspective on our construction of the models, we give some comments about the terms that are allowed by the symmetry of the theory but not included in our potentials. A canonical term in the Kähler potential is one of them. It gives rise to canonical kinetic terms for the fields. Another example is the series of inverse power terms of different powers. The inclusion of those terms could shift the VEV's by $O(1)$ factor, but we may expect the theory to remain the same qualitatively, i.e. the theory remains free of ghosts. From the analysis presented in this paper, it should not be hard to imagine that availability of ghost free theory would not be lost just by including those terms, especially given the fairly loose conditions for the coupling constants. The higher dimensional terms are not included as well. Their effects are expected to be small. For these reasons our simple example may be sufficient for a demonstration of a possibility that a consistent theory with noncompact gauge symmetry can be constructed.

We have discussed the obstacles that we encounter in the construction. One of the difficulties is related to the instability. The noncompact nature of the group generally gives rise to directions in the field space that destabilize the system. In order to construct a model with bounded potentials, we need to find a way to eliminate such directions. For F-term potentials, it seems easy to eliminate them by choosing the appropriate Kähler potentials. On the other hand, finding bounded D-term potentials by searching for positive-definite gauge kinetic function seems difficult. One way to avoid such instability is to construct a model so that these directions are related to the gauge transformations. We conjecture that this is possible when certain matter contents are chosen.

The analysis presented in this paper are confined to the study of classical properties, i.e. the leading order of perturbative expansions. One might wonder whether quantum

corrections may change them drastically and the whole arguments become invalid. However, if the perturbation theory is well-behaved, we may expect otherwise. What concerns us the most about quantum corrections is whether our assumptions are compatible with them. Since the potentials we employed are of a quite unfamiliar type (the inverse type) we do not know what to expect with certainty. To the best of our knowledge, no literature exists in which quantum corrections in such theories are studied. Therefore, in order to see whether our assumptions are consistent, it is desirable to study the nature of quantum corrections, especially the renormalization of Kähler potentials. Even if the assumption is shown to be incompatible with quantum corrections, it does not necessarily mean the breakdown of our construction. It should not be unreasonable to expect the possibility that the parameters of the theory are finely-tuned due to the dynamics of more fundamental theory in such a way that the dimensional coupling constants behave as our assumptions.

Appendix A: Parametrization of the vacuum

In this appendix, we elaborate on the choice of the parametrization of the vacuum that we used in eq.(44) and eq.(58).

1. doublet

First, we consider the case of an $SU(1,1)$ doublet Ψ . Let us start with an arbitrary configuration

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (\text{A1})$$

where ψ_1 and ψ_2 are complex. We show that there exists an $SU(1,1)$ transformation U that takes Ψ into the form

$$\Psi' = U\Psi = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad (\text{A2})$$

where u is real. This can be done by using the explicit expression for U given as

$$U = \begin{pmatrix} F & G^\star \\ G & F^\star \end{pmatrix}, \quad (\text{A3})$$

where F and G are complex valued parameters with a constraint $|F|^2 - |G|^2 = 1$. We look for a solution for F and G to the following equations,

$$u = F\psi_1 + G^*\psi_2, \quad 0 = G\psi_1 + F^*\psi_2. \quad (\text{A4})$$

It is easy to verify that the unique solution to eq.(A4) exists. It is given as

$$F = \frac{\psi_1^*}{\sqrt{|\psi_1|^2 - |\psi_2|^2}}, \quad G = \frac{-\psi_2}{\sqrt{|\psi_1|^2 - |\psi_2|^2}}. \quad (\text{A5})$$

It should be emphasized that the solution exists only for configurations that satisfy $|\psi_1|^2 - |\psi_2|^2 > 0$. For this transformation, we have

$$u = \sqrt{|\psi_1|^2 - |\psi_2|^2}. \quad (\text{A6})$$

We have used up all three of the $SU(1, 1)$ transformation parameters.

2. triplet

Next, we consider the case of an $SU(1, 1)$ triplet X_A and confirm our statement that we made about eq.(58). Let us start with an arbitrary configuration,

$$X_A = (X_1, X_2, X_3), \quad (\text{A7})$$

where X_A are complex. The transformation law for X_A under $SU(1, 1)$ can be written as

$$X_A \rightarrow D_A^B X_B, \quad (\text{A8})$$

where D_A^B is a real matrix that satisfies

$$\eta^{(3)} D^T \eta^{(3)} D = 1. \quad (\text{A9})$$

The explicit form of D is given by exponentiating the generators in the adjoint representation given in eq.(91). For convenience, we introduce the following matrices each of which corresponds to a transformation generated by H_A

$$D_A(\phi_A) = e^{i\phi_A H_A^{(3)}}, \quad A = 1, 2, 3, \quad (\text{A10})$$

where ϕ_A is real and the summation over the index A is not implied.

Our argument proceeds in three steps. First, we focus on the imaginary parts of X_A , which are introduced in section III B and denoted by Z_A . Note that the components Z_A are all real. Because the real and imaginary parts never mix with each other by the $SU(1, 1)$ transformations, we can treat them separately. We show that a transformation that takes the arbitrary configuration of $Z_A = (v_1, v_2, v_3)$ into the form $Z_A = (0, v'_2, v'_3)$ exists, i.e. the solution to the following equations exists,

$$\begin{pmatrix} 0 \\ v'_2 \\ v'_3 \end{pmatrix} = D_3(\phi_3) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad (\text{A11})$$

where $D_3(\phi_3)$ is given by

$$D_3(\phi_3) = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a = \cos \phi_3, \quad b = \sin \phi_3. \quad (\text{A12})$$

Note that a and b satisfy $a^2 + b^2 = 1$. The solution for a and b to the eq.(A11) is given as

$$a = \frac{v_2}{\sqrt{v_1^2 + v_2^2}}, \quad b = \frac{-v_1}{\sqrt{v_1^2 + v_2^2}}, \quad (\text{A13})$$

in which case we have

$$v'_2 = \sqrt{v_1^2 + v_2^2}, \quad v'_3 = v_3. \quad (\text{A14})$$

Note that there are no restrictions on v_1 and v_2 . Next, we show that there exists a transformation that takes arbitrary configuration of $Z_A = (0, v'_2, v_3)$ into the form $Z_A = (0, 0, v''_3)$, i.e. the solution to the following equations exists,

$$\begin{pmatrix} 0 \\ 0 \\ v''_3 \end{pmatrix} = D_1(\phi_1) \begin{pmatrix} 0 \\ v'_2 \\ v_3 \end{pmatrix}, \quad (\text{A15})$$

where $D_1(\phi_1)$ is given by

$$D_1(\phi_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & d \\ 0 & d & c \end{pmatrix}, \quad c = \cosh \phi_1, \quad d = \sinh \phi_1. \quad (\text{A16})$$

Note that c and d satisfy $c^2 - d^2 = 1$. It is easy to verify that an unique solution to eq.(A15) exists and is given as

$$c = \frac{v_3}{\sqrt{v_3^2 - v_2'^2}}, \quad d = \frac{-v_2'}{\sqrt{v_3^2 - v_2'^2}} \quad (\text{A17})$$

We stress that the solution exists only for configurations that satisfy $v_3^2 - v_2'^2 > 0$, which translates into $-v_1^2 - v_2^2 + v_3^2 > 0$ in the original variables. We arrive at X_A of the following form

$$X_A = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ v_3'' \end{pmatrix}, \quad v_3'' = \sqrt{-v_1^2 - v_2^2 + v_3^2}, \quad (\text{A18})$$

where u_1, u_2 , and u_3 are real. We consider further transformation by D_3 , which does not bring any changes to the imaginary part Z_A (See eq.(A15)). With the transformation $D_3(\phi_3')$ given as

$$D_3(\phi_3') = \begin{pmatrix} a' & b' & 0 \\ -b' & a' & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a' = \frac{u_2}{\sqrt{u_1^2 + u_2^2}}, \quad b' = \frac{-u_1}{\sqrt{u_1^2 + u_2^2}}, \quad (\text{A19})$$

we can finally bring it to the form

$$X_A = \begin{pmatrix} 0 \\ u_2' \\ u_3 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ v_3'' \end{pmatrix}, \quad u_2' = \sqrt{u_1^2 + u_2^2}. \quad (\text{A20})$$

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- [34] Note that the “ghost” that we are concerned here is essentially different from the Faddeev-Popov ghosts, which are introduced in the gauge-fixing procedure and can be shown to be absent from the physical Hilbert space.
- [35] The group $SU(1,1)$ considered in the present paper is the covering group of $SO(2,1)$ [33] considered in Ref. [26]. There is no essential difference between the two.
- [36] In principle there could be more than one low-energy mass scales, but we assume that there is a single low-energy scale for simplicity.
- [37] In a vacuum that respects Poincare invariance, only scalar fields are allowed to acquire the VEV’s.